433-431 (433-631) — Quiz 2
The lambda calculus
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Notes

The Greek letters 'α', 'β', and 'η', are pronounced 'alpha', 'beta', and 'eta' respectively.

Question 1 (8 minutes)

Let $\rightarrow^*$ be the multi-step reduction relation for β reduction. If e_1, e_2, e_3 are expressions in the lambda calculus, and $e_1 \rightarrow^* e_3$, and $e_2 \rightarrow^* e_3$, is it always true that $e_1 \rightarrow^* e_2$? If yes, explain why, if no, give a counter example.

Question 2 (2 minutes)

Draw this lambda calculus expression as a tree: $\lambda x.f\ x\ (f\ x)$.

Question 3 (2 minutes)

Give an example of a term which is an η normal form, but not a β normal form.

Question 4 (5 minutes)

Describe the variable capture problem of β reduction using an example. Explain *briefly* (in a couple of sentences) a strategy to solve the problem for your specific example.

Question 5 (8 minutes)

The following statement is true. There exists a non-empty subset of the lambda calculus expressions which have a normal form which can be found by *both* the applicative-order *and* the normal-order reduction strategies.

For all such expressions which have that property, is it necessarily the case that the normal-order strategy will find the normal form in fewer reduction steps than the applicative-order strategy? If yes, explain why, if no, give a counter example.

Question 6 (5 minutes)

Give a procedure (in whatever notation you like best) which will find the set of all *free variables* in a lambda calculus expression.

Question 7 (5 minutes)

Let $Y = \lambda h.\ (\lambda x\ .\ h\ (x\ x))\ (\lambda x\ .\ h\ (x\ x))$. That is, let Y be Curry's fixed point finder. Demonstrate that Y is indeed a fixed point finder for *all* expressions in the lambda calculus. Remember that Y must satisfy this equation: $Y\ e = e\ (Y\ e)$, for all lambda calculus expressions e.

Question 8 (8 minutes)

This is the simple form of the Church-Rosser theorem for confluence of the lambda calculus:

> Let e_1, e_2, e_3, e_4 be expressions in the lambda calculus. Let $\rightarrow^*$ be the multi-step reduction relation for β and η reduction, and α conversion. If $e_1 \rightarrow^* e_2$, and $e_1 \rightarrow^* e_3$, then there exists an expression e_4, such that $e_2 \rightarrow^* e_4$, and $e_3 \rightarrow^* e_4$.

Explain why this theorem implies that if an expression has a normal form, then it is unique (in other words: all lambda calculus expressions have at most one normal form).

Question 9 (6 minutes)

Consider the following lambda calculus expressions:

$$
\begin{aligned}
K &= \lambda x\ .\ (\lambda y\ .\ x) \\
L &= (\lambda x\ .\ x\ x\ y)\ (\lambda x\ .\ x\ x\ y) \\
E &= \lambda x\ .\ z\ x
\end{aligned}
$$

Suppose we have an expression constructed like so: $(K\ E)\ L$, which has a $(\beta,\ \eta)$ normal form. Show a sequence of reduction steps which transforms the expression to its normal form.

Question 10 (6 minutes)

Two lambda calculus expressions are "syntactically equal" if they have exactly the same syntax. For example $(\lambda x\ .\ x)$ is *not* syntactically equal to $(\lambda y\ .\ y)$, even though they are α convertible. Explain why syntactic equality is considered a weak form of equality.